SECTION 4.7: INVERSE TRIG FUNCTIONS

You may want to review Section 1.8 on inverse functions.

PART A: GRAPH OF $\sin^{-1} x$ (or $\arcsin x$)

Warning: Remember that f^{-1} denotes function inverse, not multiplicative inverse (or reciprocal). Usually, $f^{-1} \neq \frac{1}{f}$. In particular, $\sin^{-1} x \neq \frac{1}{\sin x}$, or $\csc x$. We **can** say that $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$. Although it is often helpful in Calculus to rewrite $\sin^n x$ as $(\sin x)^n$, this is **not** true of $\sin^{-1} x$, because -1 is **not** an exponent in that case. However, -1 **does** act as an exponent in $(\sin x)^{-1}$.

If $f(x) = \sin x$, and the domain is **R** (which is, after all, the implied domain), then f is **not** a one-to-one function, and it has no inverse **function**.

We want to define an inverse sine (or "arcsine") function $f^{-1}(x) = \sin^{-1} x$ (or $\arcsin x$). To do so, we must restrict the domain of $f(x) = \sin x$ so that it is a one-to-one function whose graph passes the HLT (Horizontal Line Test).

What should this restricted domain be? It should be an x-interval on which the $\sin x$ graph:

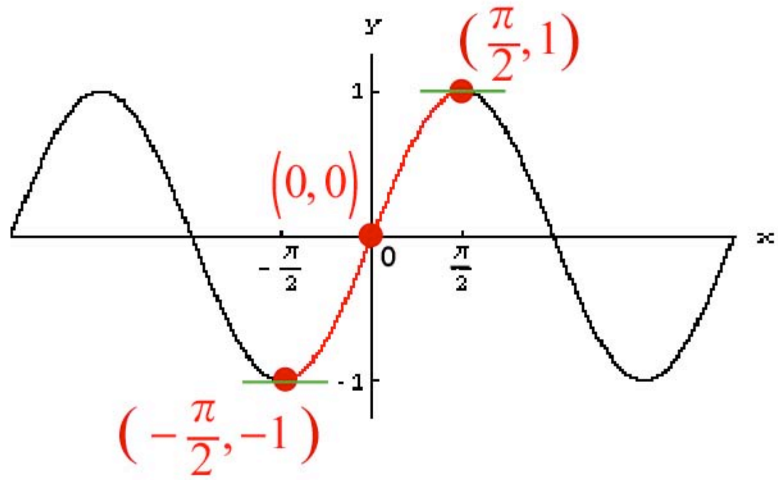
1) Passes the HLT, and

2) Is as "tall" as the original, unrestricted $\sin x$ graph. In other words, we would like the range to be the same as before.

It is universally agreed that we take the x-interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as our restricted domain.

The resulting range for our $\sin x$ function remains $[-1, 1]$.

The resulting graph is in red below:
(The *x*- and *y*-axes are scaled differently.)



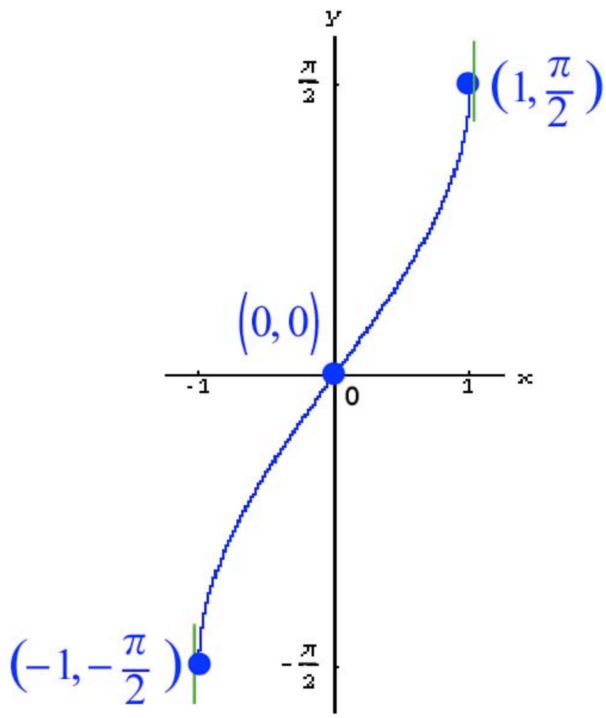
Observe that:

- The function increases on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- The graph switches from concave up to concave down at $(0, 0)$.

  It may be easier to remember that the graph is a snake of finite length that has horizontal (one-sided) tangent lines (in green) at its endpoints.

The graph of $f^{-1}(x) = \sin^{-1} x$ (or $\arcsin x$), the arcsine function, is obtained by switching the *x*- and *y*-coordinates of all the points on the red graph we just saw. (Reflecting the red graph about the line $y = x$ may be hard to visualize.) We obtain:



Observe that:

- The inverse function also increases, but on the interval $[-1, 1]$.
  The three indicated points above suggest this.

- However, the graph switches from concave down to concave up at $(0, 0)$.
  It may be easier to remember that the graph is a snake of finite length that has vertical (one-sided) tangent lines (in green) at its endpoints.

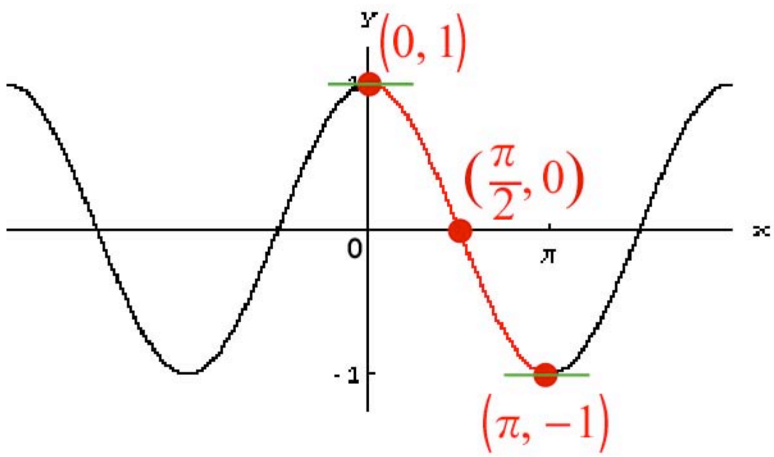
Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	**Range**
$\sin x$ (restricted)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\sin^{-1} x$ (or $\arcsin x$)	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

PART B : GRAPH OF $\cos^{-1} x$ (or $\arccos x$)

It is universally agreed that we take the x-interval $\left[0, \pi\right]$ as our restricted domain for $f\left(x\right) = \cos x$. The resulting range remains $\left[-1, 1\right]$.

The resulting graph is in red below:
(The x- and y-axes are scaled differently.)

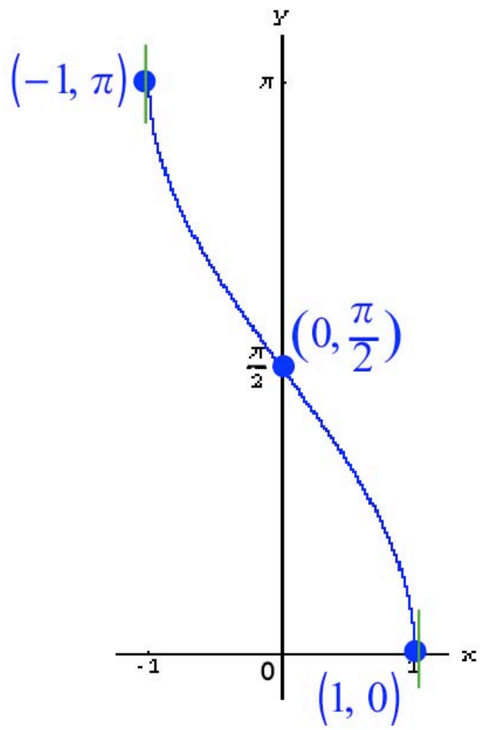


Observe that:

- The function decreases on the interval $\left[0, \pi\right]$.

- The graph switches from concave down to concave up at the "midpoint" $\left(\frac{\pi}{2}, 0\right)$. It may be easier to remember that, just as for $\sin x$, the graph is a snake of finite length that has horizontal (one-sided) tangent lines (in green) at its endpoints.

The graph of $f^{-1}(x) = \cos^{-1} x$ (or $\arccos x$), the arccosine function, is obtained by switching the x- and y-coordinates of all the points on the red graph we just saw. We obtain:



Observe that:

- The inverse function also decreases, but on the interval $[-1, 1]$.
  The three indicated points above suggest this.

- However, the graph switches from concave up to concave down at the "midpoint" $\left(0, \frac{\pi}{2}\right)$. It may be easier to remember that, just as for $\sin^{-1} x$, the graph is a snake of finite length that has vertical (one-sided) tangent lines (in green) at its endpoints.

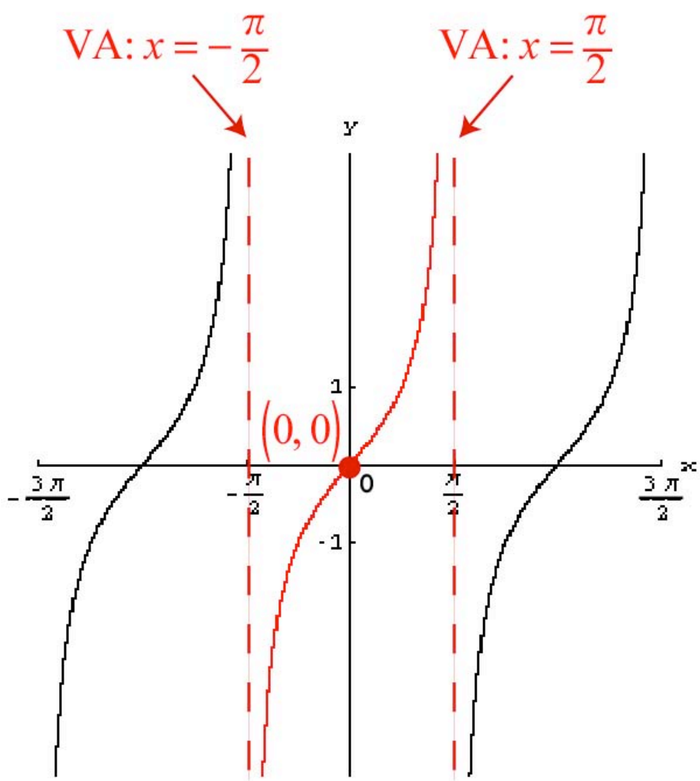
Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	**Range**
$\cos x$ (restricted)	$[0, \pi]$	$[-1, 1]$
$\cos^{-1} x$ (or $\arccos x$)	$[-1, 1]$	$[0, \pi]$

PART C: GRAPH OF $\tan^{-1} x$ (or $\arctan x$)

It is universally agreed that we take the **open** x-interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as our restricted domain for $f(x) = \tan x$. The resulting range remains **R**, or $(-\infty, \infty)$.

The resulting graph is in red below:
(The x- and y-axes are scaled differently.)

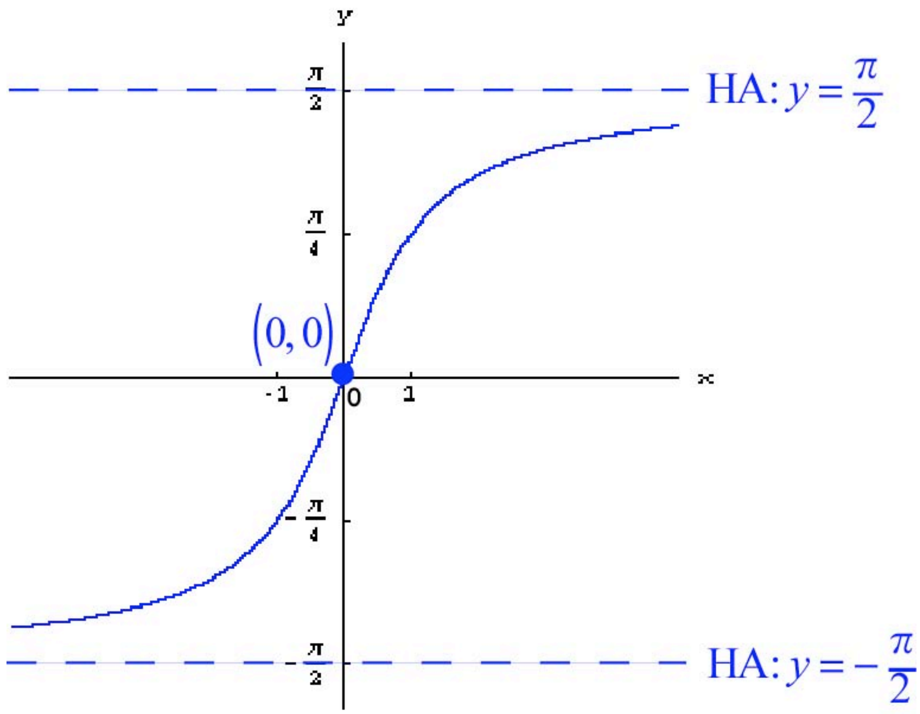


Observe that:

- The function increases on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- The graph switches from concave down to concave up at $(0, 0)$.

  It may be easier to remember that the graph is a snake of **infinite** length (unlike before) that approaches the **vertical** asymptotes (VAs) at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

The graph of $f^{-1}(x) = \tan^{-1} x$ (or $\arctan x$), the arctangent function, is obtained by switching the x- and y-coordinates of all the points on the red graph we just saw.
We obtain:



Observe that:

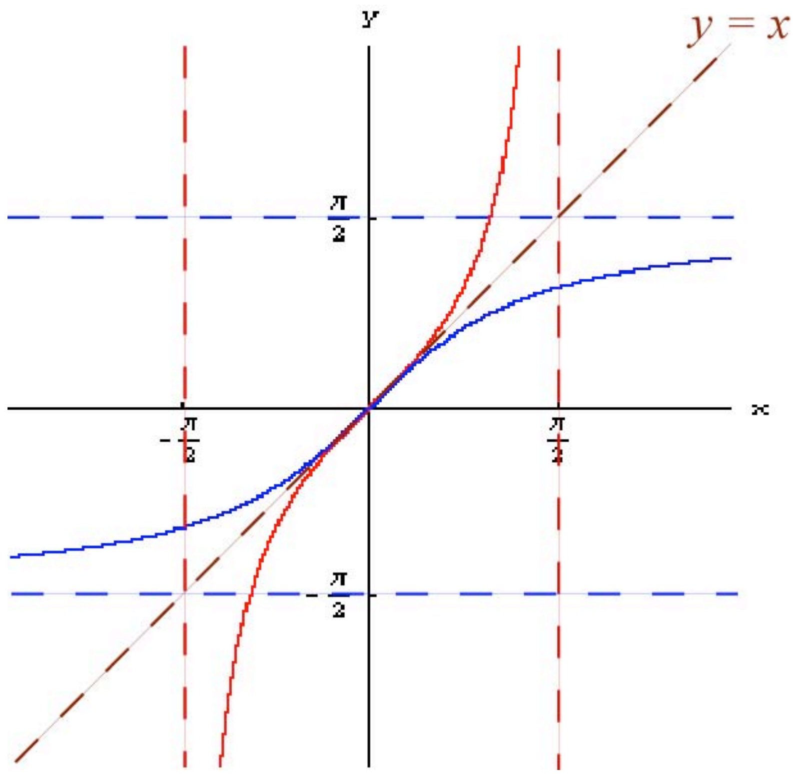
- The inverse function also increases, but on all of **R**.

- However, the graph switches from concave up to concave down at $(0, 0)$.
  It may be easier to remember that the graph is a snake of **infinite** length that approaches the **horizontal** asymptotes (HAs) at $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	**Range**
$\tan x$ (restricted)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	**R**, or $(-\infty, \infty)$
$\tan^{-1} x$ (or $\arctan x$)	**R**, or $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

You may have been able to visualize reflecting the red graph about the line $y = x$ (in brown) in order to obtain the blue graph.



Warning: The line $y = x$ may appear too flat or too steep if the x- and y-axes are scaled differently.

PART D: OTHER INVERSE TRIG FUNCTIONS

The problem with the $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ functions is that their ranges are not universally agreed upon! (In other words, there is no universal agreement about how the domains of the csc, sec, and cot functions should be restricted.) Different ranges may be used for different purposes.

For example, the range of $\sec^{-1}$ may include angles from Quadrant II (as for $\cos^{-1}$) or from Quadrant III (which tends to be more convenient in Calculus, because tan is positive there).

PART E: REMEMBERING THE RANGES OF INVERSE TRIG FUNCTIONS

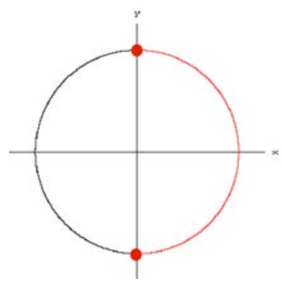
Here are some tricks:

$\sin^{-1}$

Remember that the range of $\sin^{-1}$ is the restricted domain of sin.

We could recall the red graph in Notes 4.73 and find the set of x-coordinates picked up by the graph.

We may also recall the Unit Circle. We want to focus on an arc of the Unit Circle that "picks up" **all** of the possible sin values (corresponding to y-coordinates on the circle) **exactly once** (so that we force the restricted sin function to be one-to-one). We pick the right semicircle, as opposed to the left one.

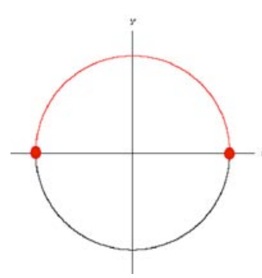


The interval of angles we typically associate with this arc is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note: We may say "angles" when "angle measures" may be more appropriate.

$\cos^{-1}$

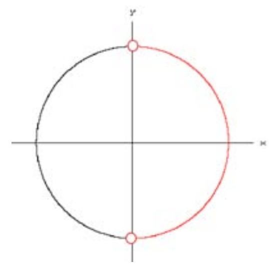
Similarly, we could recall the red graph in Notes 4.75, or we could focus on an arc of the Unit Circle that "picks up" **all** of the possible cos values (corresponding to x-coordinates on the circle) **exactly once**. We pick the top semicircle, as opposed to the bottom one.



The interval of angles we typically associate with this arc is $\left[0, \pi\right]$.

$\underline{\tan^{-1}}$

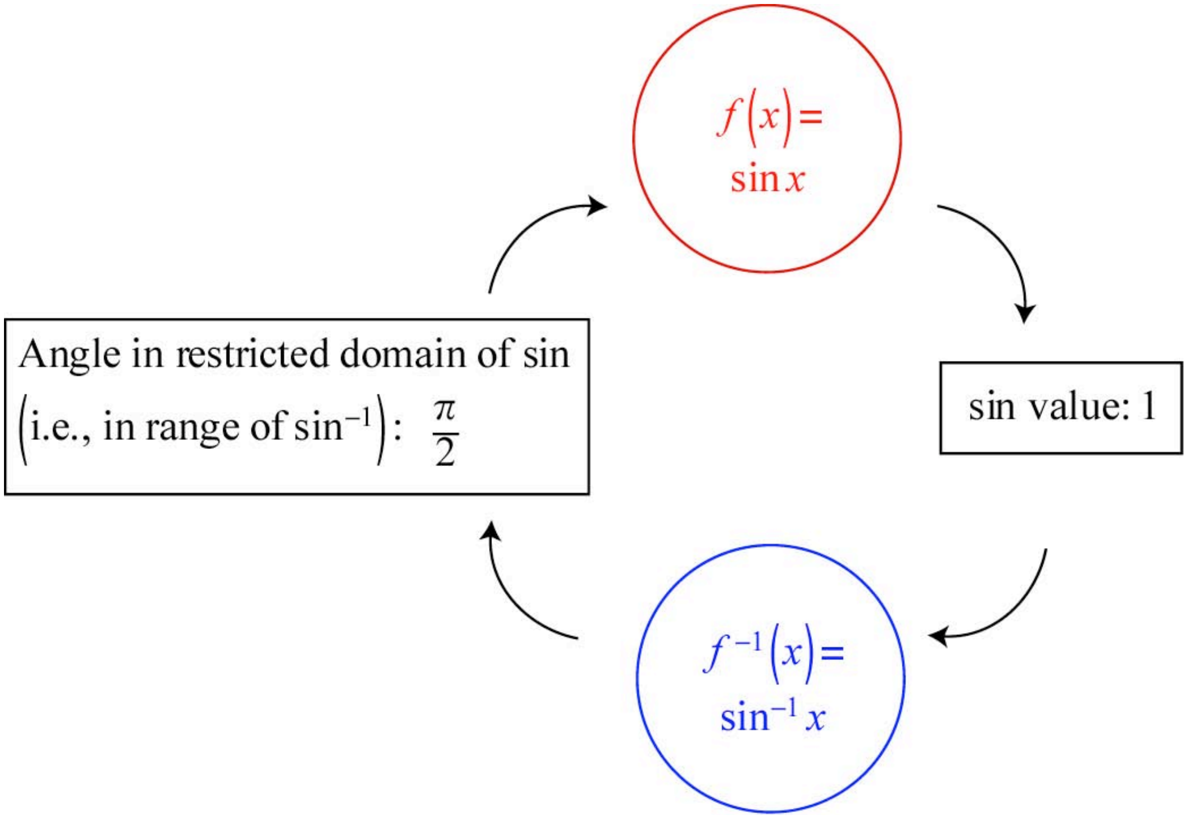
Similarly, we could recall the red graph in Notes 4.77, or we could focus on an arc of the Unit Circle that "picks up" **all** of the possible tan values (corresponding to slopes of terminal sides of standard angles) **exactly once**. As for $\sin^{-1}$, we pick the right semicircle, as opposed to the left one, but we must exclude the endpoints, because they correspond to undefined slopes.



The interval we typically associate with this arc is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

PART F: EVALUATING INVERSE TRIG FUNCTIONS

Think:



A trig function such as sin takes in angles (i.e., real numbers in its domain) as inputs and "spits out" outputs that are trig values (for sin, values between -1 and 1, inclusive).

On the other hand, an inverse trig function such as $\sin^{-1}$ takes in trig values as inputs and "spits out" angles as outputs. These angles must be in the range.

Warning: Although calculators can provide $\sin^{-1}$ values and other inverse trig values using degree measure, it is conventional to use radian measure, instead, since they directly correspond to "real numbers."

Example

Evaluate $\sin^{-1}(1)$, or $\arcsin 1$.

Solution

We know that $\sin\left(\frac{\pi}{2}\right) = 1$.

Although there are other angles whose sine is 1, $\frac{\pi}{2}$ is the only one that is a "legal" output of $\sin^{-1}$, because it is the only one in the range of $\sin^{-1}$, namely $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $\sin^{-1}(1) = \boldsymbol{\frac{\pi}{2}}$.

Example

Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, or $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$.

Solution

What angle in the range of $\sin^{-1}$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has a sin value of $-\frac{\sqrt{2}}{2}$?

Observe that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so we would like a brother of $\frac{\pi}{4}$ in Quadrant IV.

Answer: $\boldsymbol{-\frac{\pi}{4}}$.

Observe that the coterminal angle $\frac{7\pi}{4}$, for example, is not in the range.

<u>Example</u>

Evaluate $\sin^{-1}(3)$, or $\arcsin 3$.

<u>Solution</u>

This is **undefined**, because 3 is not a sin value for any angle.
Observe that 3 is not in the domain of $\sin^{-1}$, so it is an "illegal" input.

<u>Example</u>

Evaluate $\cos^{-1}\left(-\frac{1}{2}\right)$, or $\arccos\left(-\frac{1}{2}\right)$.

<u>Solution</u>

What angle in the range of $\cos^{-1}$, $[0, \pi]$, has a cos value of $-\frac{1}{2}$?

Observe that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so we would like a brother of $\frac{\pi}{3}$ in Quadrant II.

Answer: $\boldsymbol{\frac{2\pi}{3}}$.

<u>Example</u>

$\tan^{-1}(10) \approx 1.47$ [radians]. Observe that $\tan^{-1}(10)$ is **not** undefined, because its domain is **R**; any real number is a slope (or tan value).

PART G: INVERSE PROPERTIES

Inverse Properties: Group 1

If x is an appropriate trig (i.e., sin, cos, tan) value, then:

$$\sin\left(\sin^{-1} x\right) = x \quad \left(\text{if } x \text{ is in } \left[-1,1\right]\right)$$
$$\cos\left(\cos^{-1} x\right) = x \quad \left(\text{if } x \text{ is in } \left[-1,1\right]\right)$$
$$\tan\left(\tan^{-1} x\right) = x \quad \left(\text{if } x \text{ is in } \mathbf{R}\right)$$

Otherwise, we have "**undefined**."

Think: "Unwrapping," or "undoing."

Examples

$$\cos\left(\underbrace{\cos^{-1}(0.2)}_{\substack{\text{an angle whose} \\ \text{cosine is 0.2}}}\right) = \mathbf{0.2}$$

$$\cos\left(\underbrace{\cos^{-1}(10)}_{\text{undefined}}\right) \text{ is } \textbf{undefined}$$

$$\tan\left(\underbrace{\tan^{-1}(10)}_{\substack{\text{an angle whose} \\ \text{tangent is 10}}}\right) = \mathbf{10}$$

Inverse Properties: Group 2

If θ is in the range of the appropriate inverse trig function, then:

$$\sin^{-1}(\sin\theta)=\theta \quad \left(\text{if } \theta \text{ is in } \left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$$

$$\cos^{-1}(\cos\theta)=\theta \quad \left(\text{if } \theta \text{ is in } [0,\pi]\right)$$

$$\tan^{-1}(\tan\theta)=\theta \quad \left(\text{if } \theta \text{ is in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

Example

$$\sin^{-1}\left(\sin\frac{\pi}{10}\right)=\boldsymbol{\frac{\pi}{10}}$$

Think: The $\frac{\pi}{10}$ angle looks over at $\sin^{-1}$ and asks, "Can you spit me out?" The $\sin^{-1}$ function says, "Yes, I can, because you are in my range." Also, sin says, "$\frac{\pi}{10}$ is in my domain, so I'm OK with that."

Note: The "unwrapping" properties described in the box above always work for acute angles θ such as $\frac{\pi}{10}$.

Example

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right)$$
$$= \boldsymbol{\frac{\pi}{6}}$$

“Unwrapping” doesn’t work here, because $\frac{5\pi}{6}$ is **not** in the range of $\sin^{-1}$.

The especially talkative $\sin^{-1}$ says to $\frac{5\pi}{6}$, “I can’t spit you out, but I can spit out your brother.” “When in doubt, work it out,” or … observe that we are looking for a brother of $\frac{5\pi}{6}$ that also has a sin value of $\frac{1}{2}$ and that lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because $\frac{1}{2} > 0$, we look in Quadrant I.

The brother we want is $\frac{\pi}{6}$.

PART H: USING RIGHT TRIANGLES TO WRITE ALGEBRAIC EXPRESSIONS

Example

Write $\cot\left(\cos^{-1}\frac{x}{3}\right)$ as an equivalent algebraic expression in x.

Assume that x is such that all relevant trig and inverse trig values are defined.

Solution

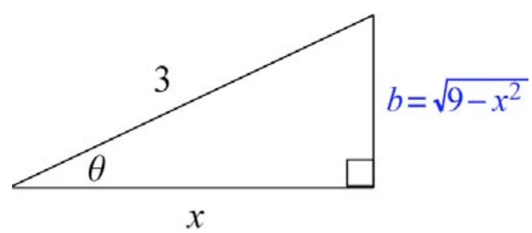
Let the angle $\theta = \cos^{-1}\left(\frac{x}{3}\right)$. Think: $\cot\left(\underbrace{\cos^{-1}\frac{x}{3}}_{=\theta}\right)$

This is true $\Leftrightarrow$ $\cos\theta = \frac{x}{3}$ and θ is in $\left[0,\pi\right]$, the range of $\cos^{-1}$.

We may actually assume that θ is acute, without loss of generality.
(If you are dealing with $\csc^{-1}$, $\sec^{-1}$, or $\cot^{-1}$, define the ranges carefully.)

Technical Note: This is a nontrivial observation! See Stewart's Precalculus book for more details.

Construct a model right triangle such that $\cos\theta = \frac{x}{3}$.



Use the Pythagorean Theorem to find an expression for the missing side length, b.

$$x^2 + b^2 = 9$$
$$b^2 = 9 - x^2$$
$$b = \pm\sqrt{9 - x^2}$$
$$b = \sqrt{9 - x^2} \quad \left(\text{Take the "+" root.}\right)$$

We see that $\tan\theta = \frac{\sqrt{9-x^2}}{x}$.

Warning: $\sqrt{9-x^2} \neq 3-x$.

We then see that $\cot\theta = \frac{x}{\sqrt{9-x^2}}$, or $\frac{x\sqrt{9-x^2}}{9-x^2}$, which is our answer.

Note: Some books don't require that you rationalize the denominator.

In Calculus: This technique is used when you perform integration using trigonometric substitutions. You will see this in Calculus II: Math 151 at Mesa.